A Level set method without re-initialization

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ABSTRACT

Moving interface problems have practical applications in the fields of fluid mechanics, solid mechanics or medical imaging. There are several numerical methods in CFD to solve these problems. The volume of fluid method (VOF) of Hirt and Nichols set in 1981 is widely used for cases where the interface separates the domain into two distinct areas as in the case of two-phase flows [1]. The mixture model is based on the principle of averaged equations. The method of averaging is a wise choice to do [2] when the scale of one of the phases is too small compared to the others [3, 4]. The Level set method was first introduced by Osher and Sethian [5] to capture a moving front. This method was originally developed for the simulation of a phase change problem governed by a diffusion equation. Other applications followed in image analysis [6]. Earlier, the Level set equation was solved with the method of finite differences. Recently, a new variational formulation has been developed in order to remove the re-initialization process which is a reset phase [7]. Moreover, in most cases we are interested in having greater accuracy at the interface and not in the entire domain. Hence, the Level set method restricted to a narrow band around the zero level set was developed [8].

In this paper, a new stabilized finite element formulation is introduced to solve the level set equation without re-initialization. This method is compared with the one introduced in [7] on a time-reversed flow field case [10].

1. THE LEVEL SET METHOD

The Level set method was introduced in 1988 by Osher and Sethian [5]. The starting point of this method is the definition of a level set scalar function $\phi$. The zero value of the level set function is the interface that is transported by the velocity field. The contours of the level set function initialized as a distance function may move away from the level set distance function due to the accumulation of numerical errors, hence the need to reset the solution after a number of steps. The moving interface $\Gamma$ is the zero-level for the scalar function $\phi(t, x)$: $\Gamma(t) = \{x \in \mathbb{R}^2 | \phi(t, x) = 0\}$.

For example in a two-phase flow the domain $\Omega = \Omega_1 \cup \Omega_2$ is divided into two subdomains using the sign of the level set function:

$$
\begin{align*}
\phi(t, x) &< 0 \text{ if } x \in \Omega_1 \\
\phi(t, x) &= 0 \text{ if } x \in \Gamma \\
\phi(t, x) &> 0 \text{ if } x \in \Omega_2
\end{align*}
$$

Figure 1: Two-phase domain

Most often the level set function is initialized as the signed distance function:

$$
\begin{align*}
\phi(t, x) &= -d(t, x) \text{ if } x \in \Omega_1 \\
\phi(t, x) &= 0 \text{ if } x \in \Gamma \\
\phi(t, x) &= d(t, x) \text{ if } x \in \Omega_2
\end{align*}
$$

with

$$
\begin{align*}
d(t, x) &= \min \limits_{x \in \Gamma} |x - x_f| \text{ if } x \in \Omega
\end{align*}
$$

The signed distance function has the following useful property:

$$
|\nabla d| = 1
$$
The level set function evolution equation is:
$$\frac{\partial \phi}{\partial t} + F|\nabla \phi| = 0$$
In general case, this last equation is called the Hamilton-Jacobi equation.
For our case, the function $F$ of this previous equation is replaced by the relation $F = \bar{u} \cdot \nabla \phi$, because the level set function $\phi$ is transported by the velocity field $\bar{u}$.
Similarly, the total area must be conserved for both phases:
$$\frac{D\Omega_1}{Dt} = 0 \text{ if } \phi(t, x) < 0$$
$$\frac{D\Omega_2}{Dt} = 0 \text{ if } \phi(t, x) > 0$$

2. THE RE-INITIALIZATION METHOD

The transport equation of the level set function is solved for a few time steps. If the level set function is no longer a distance function because of errors accumulation, then an equation of reset proposed by Sussman and Fatemi [9] is solved to correct the signed distance in the field:
$$\frac{\partial \phi}{\partial t} + \text{sign}(\phi_0)(|\nabla \phi| - 1) = 0$$
where $\phi$ is the corrected level set function, $\phi_0$ is the function defined by $\phi_0 = \phi(t = 0, x)$ and the sign function is:
$$\text{sign}(x) = \begin{cases} 
1 \text{ if } x > 0 \\
0 \text{ if } x = 0 \\
-1 \text{ if } x < 0
\end{cases}$$
The accuracy of this method depends on the value of time between two re-initializations steps. The level set function initialized as a signed distance function do not stay as a signed distance function during the time simulation. The time after which the level set solution is re-initialized, is chosen in a suitable manner. However, the re-initialization process can move the zero level set contours from its initial position according to Sethian [6]. When the level set function $\phi(t, x)$ is too far from the signed distance function $d(x)$, this method is not able to re-initialize the solution $\phi(t, x)$.

3. THE VARIATIONAL METHOD BY ENERGY PENALIZATION OF LI-XU- GUI-FOX

In this method [7], we consider the functional:
$$\mathcal{P}(\phi) = \int_\Omega \frac{1}{2} (|\nabla \phi| - 1)^2 \, d\Omega$$
This leads to define an energy:
$$\mathcal{E}(\phi) = \lambda \mathcal{P}(\phi) + \mathcal{E}_m(\phi)$$
where $\lambda$ is a parameter which controls the deviation between the level set function $\phi$ and the signed distance function $d(x)$.

The problem is to minimize the total energy. As a result, the transport problem of the penalized level set function becomes:
$$\frac{\partial \phi}{\partial t} + \bar{u} \cdot \nabla \phi - \lambda_1 \text{div}(K \nabla \phi) = 0$$
where $K = 1 - \frac{1}{|\nabla \phi|}$.

This equation is a transport-diffusion equation when $K > 0$ (that is to say that $|\nabla \phi| > 1$). In contrary, it is an anti-diffusion-transport equation when $K < 0$ (that is to say that $|\nabla \phi| < 1$).

This method has been implemented in a finite element code. In what follows we summarize the variational formulation.

Remark 1: In reference to Li, Xu, Gui and Fox [7], the penalty coefficient $\lambda$ is a constant to be determined empirically. Note that penalty coefficient $\lambda$ has the dimension of diffusivity. A too high value of this coefficient will affect the transport problem. To take into account the effect elements size and the intensity of the speed, we propose a variable coefficient using the following formula which is widely used in the stabilization methods:
$$\lambda_1 = \beta \frac{|\|u\|_h|}{2}$$
where $\beta$ is the size of the element, $\|u\|$ the magnitude of the velocity and $\lambda$ an empirical constant which has a small acceptable range. We found an optimal value of $\beta$ is between 0.02 and 0.05.

Remark 2: Although there is a penalty term for the constraint $|\nabla \phi| - 1 = 0$, this constraint is satisfied neither locally nor globally. But the penalty factor helps to tend the solution to the constraint.

4. A STABILIZED VARIATIONAL METHOD

We propose a new variational formulation to constrain the level set function to remain a distance function during the transportation by the velocity field (at least in the weak sense).

The transport equation is solved with the constraint $|\nabla \phi| - 1 = 0$ which is a non-linear equation. A Lagrange multiplier function $\psi$ is then introduced.

The stabilized variational formulation find $(\phi, \psi)$ such that for all $(\psi, q)$:
$$\int_\Omega \left[ (\psi + \tau \bar{u} \cdot \nabla \phi) \left( \frac{\partial \phi}{\partial t} + \bar{u} \cdot \nabla \phi \right) + \lambda_2 (|\nabla \phi| \cdot \nabla \psi) p ight] \, d\Omega$$
$$+ \int_\Omega \lambda_1 K (\nabla \phi \cdot \nabla \psi) \, d\Omega$$
$$+ \lambda_2 \int_\Omega q (|\nabla \phi|^2 - 1) \, d\Omega = 0$$
with \( \tau = \alpha \left( \frac{h}{\|u\|} + \frac{\Delta t}{2} \right) \) where \( \Delta t \) is the time step and \( \alpha = 0 \) or 1. For \( \alpha = \lambda_1 = \lambda_2 = 0 \) we find the standard Galerkin formulation. For \( \alpha = 1 \) and \( \lambda_1 = \lambda_2 = 0 \) we find the stabilized SUPG formulation. For \( \alpha = 1 = \lambda_1 \) and \( \lambda_2 = 0 \) we find the stabilized penalized formulation of [5]. Finally, for \( \alpha = \lambda_1 = \lambda_2 = 1 \) we find the stabilized formulation with the Lagrange multiplier.


This is a well-documented test case used to assess level set methods. A single vortex flow is defined by the stream function:

\[
\Psi_{\text{single vortex}} = \frac{1}{\pi} \sin^2(\pi x) \sin^2(\pi y)
\]

The time-reversed vortex flow is defined by multiplying the single vortex flow by \( \cos \left( \frac{\pi t}{T} \right) \), where \( T \) is the period.

\[
\Psi_{\text{time-reversed vortex}} = \frac{1}{\pi} \sin^2(\pi x) \sin^2(\pi y) \cos \left( \frac{\pi t}{T} \right)
\]

The relation between the velocity and the stream function is:

\[
u = \frac{\partial \psi}{\partial x} \quad \text{and} \quad u = \frac{\partial \psi}{\partial y}
\]

Then,

\[
\begin{align*}
\psi_{\text{time-reversed vortex}} &= -2\sin^2(\pi x) \cos(\pi y) \sin(\pi y) \cos \left( \frac{\pi t}{T} \right) \\
\psi_{\text{time-reversed vortex}} &= 2\cos(\pi x) \sin(\pi x) \sin^2(\pi y) \cos \left( \frac{\pi t}{T} \right)
\end{align*}
\]

Triangular elements are used with quadratic finite elements approximations for the variable \( \phi \) and a constant per element approximation for the variable \( p \). For time discretization a second-order Gear scheme is used. The value of the time step \( \Delta t \) for each mesh is in Table 1.

A circle separates the unit square computational domain into two subdomains. The circle is centered at \((0.5, 0.75)\), and its radius is 0.15.

The different meshes used for the analysis and the interface at the initial position are shown in figures 2, 3, and 4.
The simulation is done for a period $T$ of 8s. The flow reverses after the half period.

The results obtained with the different meshes with the standard Galerkin finite element method for $t = 0$ s, 2 s, 4 s, 6 s, and 8 s are shown in the figures 5, 6, and 7.

At the end of the simulation, the phase which is located in the circle is distorted from its original position and the mass loss is decreasing with the mesh refinement. The values of the mass loss for both the different cases and different meshes are shown in Table 2. The mass loss is noticeable at the
trailing edge of the spiral which was the disk domain. The values of the parameters that are used are enumerated in Legend 1:

Case 1: \( \alpha = 1, \lambda_1 = 0 \) and \( \lambda_2 = 1 \)
Case 2: \( \alpha = 1, \beta = 0.02 \) and \( \lambda_2 = 0 \)
Case 3: \( \alpha = 1, \lambda_1 = 0 \) and \( \lambda_2 = 0 \)
Case 4: \( \alpha = 0, \lambda_1 = 0 \) and \( \lambda_2 = 0 \)

Legend 1

The mass loss of the disk is computed with the formula:

\[
\text{% mass loss of the disk} = \frac{\text{area}_{\text{domain computed}} - \text{area}_{\text{domain exact}}}{\text{area}_{\text{domain exact}}} \times 100
\]

The results of mass loss of the disk obtained for the different formulation are listed below.

<table>
<thead>
<tr>
<th>Case</th>
<th>Mesh size</th>
<th>32X32</th>
<th>64X64</th>
<th>128X128</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.0</td>
<td>-100.0</td>
<td>-35.64</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-100.0</td>
<td>-79.61</td>
<td>-35.70</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-72.04</td>
<td>-7.51</td>
<td>1.92</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-60.53</td>
<td>-6.19</td>
<td>1.75</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Mass loss of the disk

For all of the cases, the mass loss decreases with the mesh refinement. Moreover, for some cases the disk totally disappears before the end of the simulation especially for the coarse mesh. Very surprisingly, the standard Galerkin method offers the best results. The stabilization methods add more diffusion.

We define the following norms and errors:

\[
\text{Error } L_1 = \| \phi_{\text{exact}} - \phi_{\text{computed}} \|_1
\]

\[
\text{Error } L_1 = \int_{\Omega} | \phi_{\text{exact}} - \phi_{\text{computed}} | \, dx \, dy
\]

\[
\text{Error } L_2 = \| \phi_{\text{exact}} - \phi_{\text{computed}} \|_2
\]

\[
\text{Error } L_2 = \sqrt{\int_{\Omega} (\phi_{\text{exact}} - \phi_{\text{computed}})^2 \, dx \, dy}
\]
The values of these norms for different meshes are presented in the following figures and for:

Case 3: \( \alpha = 1, \lambda_1 = 0 \) and \( \lambda_2 = 0 \)
Case 4: \( \alpha = 0, \lambda_1 = 0 \) and \( \lambda_2 = 0 \)

Legend: 2

Figures 12 to 16 show that the errors are minimal when the standard Galerkin method is used.
CONCLUSION

In the present article, a new variational formulation is proposed to constrain the level set function to stay as a distance function during the computation. The method also doesn’t require re-initialization. It can be seen as a generalization of the method proposed by Osher and Sethian [5]. The discretization of the problem is done in time with a second-order Gear scheme and in space with triangular elements using quadratic finite elements approximations. In this paper, the method is applied to a reversed vortex case. The comparison is done between the initial and the final positions of the interface which should be the same. Remarkably, the best results are obtained with the standard Galerkin formulation. Therefore, for this well-known test case, a standard Galerkin formulation with second order space and time discretization provides good results. The stabilized formulations seem to add more diffusion causing mass losses. More numerical experiments for complex flows (computed from a Navier-Stokes solver) will be shown in the presentation.

REFERENCES